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A generalized duality transformation of the anisotropic XY chain in a magnetic field

Haye Hinrichsen
Physikalisches Institut, Universität Bonn, Nussallee 12, D-53115 Bonn, Federal Republic of Germany

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Abstract. We consider the anisotropic XY chain in a magnetic field with special boundary conditions described by a two-parameter Hamiltonian. It is shown that the exchange of the parameters corresponds to a similarity transformation, which reduces in a special limit to the Ising duality transformation.

In this paper we consider the anisotropic XY chain in a magnetic field which is defined by the Hamiltonian

$$H^{XY}(\eta, q) = -\tfrac{1}{2}\sum_{j=1}^{L-1}\left(\eta\sigma_j^x\sigma_{j+1}^x + \eta^{-1}\sigma_j^y\sigma_{j+1}^y + q\sigma_j^z + q^{-1}\sigma_{j+1}^z\right) \tag{1}$$

where q and η are complex parameters and $\sigma_j^{x,y,z}$ are Pauli matrices acting on site j. Up to boundary terms, which will play a crucial rôle here, H can be rewritten as

$$H^{XY}(\eta, h) = -\tfrac{1}{2}\sum_{j=1}^{L}\left(\eta\sigma_j^x\sigma_{j+1}^x + \eta^{-1}\sigma_j^y\sigma_{j+1}^y\right) - h\sum_{j=1}^{L}\sigma_j^z \tag{2}$$

where $h = (q + q^{-1})/2$ is the magnetic field. This Hamiltonian has a long history [1,2] and provides a good model for helium adsorbed on metallic surfaces (η real and q on the unit circle). It also gives the master equation of the kinetic Ising model [3] ($q = 1$ and η real) and plays a role in one-dimensional reaction–diffusion processes [4]. For the special boundary conditions defined in (1) the chain has been shown to be invariant under a two-parameter deformation of the $su(1|1)$ superalgebra [5], and some of their correlation functions in the massless regime have been computed in [6].

The aim of this paper is to show that for these boundary conditions the exchange of the parameters q and η in the Hamiltonian (1) corresponds to a similarity transformation

$$H^{XY}(\eta, q) \doteq H^{XY}(q, \eta) \tag{3}$$

which reduces in a special limit to the Ising duality transformation (here '$\doteq$' denotes equality up to a similarity transformation). The Ising limit of the XY chain is given by

$$H^{\mathrm{IS}}(a, b) = \lim_{\xi\to\infty}(1/\xi)H^{XY}(a\xi, b\xi) \tag{4}$$

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where

$$H^{\mathrm{IS}}(a,b) = -\tfrac{1}{2}\sum_{j=1}^{L-1}\left(a\sigma_j^x\sigma_{j+1}^x + b\sigma_j^z\right) \tag{5}$$

is the Ising Hamiltonian with mixed boundary conditions [7]. One of the most remarkable properties of the Ising model is its self-duality [8]. For the boundary conditions defined in (5) the Ising duality transformation

$$\sigma_j^x \to \tilde{\sigma}_j^x = \prod_{i=1}^{j}\sigma_i^z \qquad \sigma_j^z \to \tilde{\sigma}_j^z = \sigma_j^x\sigma_{j+1}^x \tag{6}$$

takes place as

$$H^{\mathrm{IS}}(a,b) \doteq H^{\mathrm{IS}}(b,a) + a(\sigma_L^z - \sigma_1^z). \tag{7}$$

Using (4) we can rewrite (7) as

$$\lim_{\xi\to\infty}(1/\xi)H^{XY}(a\xi, b\xi) \doteq \lim_{\xi\to\infty}(1/\xi)H^{XY}(b\xi, 1/a\xi). \tag{8}$$

Notice that we absorbed the surface terms in (7) by inserting the argument $1/a\xi$ instead of $a\xi$ on the RHS of (8). In order to symmetrize this expression, we perform a rotation

$$\sigma_j^x \to \sigma_j^y \qquad \sigma_j^y \to -\sigma_j^x \qquad \sigma_j^z \to \sigma_j^z \qquad (j = 1,\ldots,L) \tag{9}$$

on the LHS of (8)

$$H^{XY}(a\xi, b\xi) \doteq H^{XY}(1/a\xi, b\xi) \tag{10}$$

and we obtain

$$\lim_{\xi\to\infty}(1/\xi)H^{XY}(1/a\xi, b\xi) \doteq \lim_{\xi\to\infty}(1/\xi)H^{XY}(b\xi, 1/a\xi). \tag{11}$$

This means that (3) holds for $\eta = 1/a\xi$ and $q = b\xi$ in the limit $\xi \to \infty$.

The aim of this paper is to prove (3), i.e. we derive a similarity transformation

$$H^{XY}(\eta, q) = UH^{XY}(q, \eta)U^{-1} \tag{12}$$

for *arbitrary* parameters η and q. Let us first summarize our results. For this purpose let us introduce fermionic operators by a Jordan–Wigner transformation

$$\tau_j^{x,y} = \left(\prod_{i=1}^{j-1}\sigma_i^z\right)\sigma_j^{x,y} \tag{13}$$

which allows the Hamiltonian (1) to be written as

$$H(\eta, q) = \frac{\mathrm{i}}{2}\sum_{j=1}^{L-1}\left(\eta\tau_j^2\tau_{j+1}^1 - \eta^{-1}\tau_j^1\tau_{j+1}^2 + q\tau_j^1\tau_j^2 + q^{-1}\tau_{j+1}^1\tau_{j+1}^2\right). \tag{14}$$

Denoting

$$\alpha = \frac{q}{\eta} \qquad \omega = \left(\frac{\alpha^{1/2} - \alpha^{-1/2}}{\alpha^{1/2} + \alpha^{-1/2}}\right) \tag{15}$$

the explicit expression for $U(\alpha)$ is given by the polynomial

$$U(\alpha) = \frac{1}{\sqrt{N}} \sum_{k=0}^{[L/2]} \omega^k G_{2k} \tag{16}$$

where the generators G_{2k} are defined by

$$G_n = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant L} \tau_{j_1}^x \tau_{j_2}^x \cdots \tau_{j_n}^x. \tag{17}$$

By convention we take $G_0 \equiv \mathbf{1}$, and $[L/2]$ denotes the truncation of $L/2$ to an integer number. N is a normalization constant which is given by

$$N = \sum_{k=0}^{[L/2]} \binom{L}{2k} \omega^{2k} = 2^{L-1} \frac{1+\alpha^L}{(1+\alpha)^L}. \tag{18}$$

Notice that the transformation depends only on the ratio $\alpha = q/\eta$. Obviously the normalization N vanishes for $\alpha^L = -1$ so that the transformation (12) diverges, and therefore we will exclude this case in the following. For $\alpha = 1$ the transformation $U(\alpha)$ reduces to the identity, and this is what we expect since for $\eta = q$ the exchange of η and q does not effect a change in the Hamiltonian (1).

In order to express the transformation in a formal way, let us introduce the 'time-ordered product'

$$T\tau_i^x \tau_j^x = \begin{cases} \tau_i^x \tau_j^x & i < j \\ -\tau_j^x \tau_i^x & i > j \\ 0 & i = j \end{cases} \tag{19}$$

which arranges the operators τ_j^x in increasing order with respect to their fermionic commutation relations. Observing that

$$G_{2k} = \frac{1}{k!} T G_2^k \tag{20}$$

where

$$G_2 = \sum_{1 \leqslant j_1 < j_2 \leqslant L} \tau_{j_1}^x \tau_{j_2}^x \tag{21}$$

we can rewrite (16) formally as a time-ordered exponential of G_2

$$U(\alpha) = \frac{1}{\sqrt{N}} T \exp(\omega G_2). \tag{22}$$

This expression suggests that the inverse of $U(\alpha)$ is obtained by taking $\omega \rightarrow -\omega$, i.e. $\alpha \rightarrow \alpha^{-1}$. In fact, one can show that

$$U^{-1}(\alpha) = U(\alpha^{-1}). \tag{23}$$

On the other hand we observe that $G_2^T = -G_2$ and thus the transformation (12) is an orthogonal one

$$U^T(\alpha) = U^{-1}(\alpha). \tag{24}$$

It is interesting to know how the Pauli matrices change under the transformation

$$\tilde{\sigma}_j^{x,y,z} = U(\alpha)\sigma_j^{x,y,z}U^{-1}(\alpha). \tag{25}$$

As we are going to show below, one obtains three important identities

$$\eta\tilde{\sigma}_j^x\tilde{\sigma}_{j+1}^x + q\tilde{\sigma}_j^z = q\sigma_j^x\sigma_{j+1}^x + \eta\sigma_j^z \tag{26}$$

$$q\tilde{\sigma}_j^y\tilde{\sigma}_{j+1}^y + \eta\tilde{\sigma}_{j+1}^z = \eta\sigma_j^y\sigma_{j+1}^y + q\sigma_{j+1}^z \tag{27}$$

$$\tilde{\sigma}_j^x\tilde{\sigma}_{j+1}^y = \sigma_j^x\sigma_{j+1}^y. \tag{28}$$

Because of these identities we have

$$\begin{aligned}\tilde{H}^{XY}(q,\eta) &= -\frac{1}{2}\sum_{j=1}^{L-1}[q\tilde{\sigma}_j^x\tilde{\sigma}_{j+1}^x + q^{-1}\tilde{\sigma}_j^y\tilde{\sigma}_{j+1}^y + \eta\tilde{\sigma}_j^z + \eta^{-1}\tilde{\sigma}_{j+1}^z]\\ &= -\frac{1}{2}\sum_{j=1}^{L-1}[\eta\sigma_j^x\sigma_{j+1}^x + \eta^{-1}\sigma_j^y\sigma_{j+1}^y + q\sigma_j^z + q^{-1}\sigma_{j+1}^z]\\ &= H^{XY}(\eta, q)\end{aligned} \tag{29}$$

and our claim in (3) is proved. The identities (26)–(28) contain even more information; since they hold independently for every $1 \leqslant j < L$, it is obvious that even the spectrum of the Hamiltonian

$$\bar{H} = -\tfrac{1}{2}\sum_{j=1}^{L-1}\left[a_j(\eta\sigma_j^x\sigma_{j+1}^x + q\sigma_j^z) + b_j(\eta^{-1}\sigma_j^y\sigma_{j+1}^y + q^{-1}\sigma_{j+1}^z) + c_j\sigma_j^x\sigma_{j+1}^y\right] \tag{30}$$

is invariant under the exchange $q \leftrightarrow \eta$ for *arbitrary* coefficients a_j, b_j and c_j.

We are now going to derive the identities (26)–(28). For this purpose we first consider the transformation properties of the fermionic operators $\tilde{\tau}_j^{x,y} = U\tau_j^{x,y}U^{-1}$. It turns out that

$$\tilde{\tau}_i^x = \sum_{i=1}^{L} u_{i,j}\tau_j^x \qquad \tilde{\tau}_j^y = \tau_j^y \tag{31}$$

where

$$u_{i,j} = \begin{cases} \varrho & \text{if } i = j \\ (\varrho - \alpha)\alpha^{i-j} & \text{if } i < j \\ (\varrho - \alpha^{-1})\alpha^{i-j} & \text{if } i > j \end{cases} \tag{32}$$

and

$$\varrho = \frac{\alpha^{(L/2)-1} + \alpha^{(-L/2)+1}}{\alpha^{L/2} + \alpha^{-L/2}}. \tag{33}$$

Notice also that the transformation (31) is an orthogonal one ($\sum_{k=1}^{L} u_{i,k}u_{j,k} = \delta_{i,j}$) and for $\alpha = 1$ reduces to the identity transformation $u_{i,j} = \delta_{i,j}$. Furthermore the coefficients $u_{i,j}$ depend only on the difference of their indices $i - j$. Obviously τ_j^y is invariant under the similarity transformation, and this immediately proves (28). By adding the Jordan–Wigner transformation (13) we obtain the following transformation rules for the other terms occuring in the Hamiltonian

$$\tilde{\sigma}_j^x \tilde{\sigma}_{j+1}^x = (\varrho - \alpha^{-1}) \sum_{k=1}^{j-1} \alpha^{j-k+1} \sigma_k^y S_{k+1,j-1} \sigma_j^y + (\varrho - \alpha) \sum_{k=j+2}^{L} \alpha^{j-k+1} \sigma_j^x S_{j+1,k-1} \sigma_k^x + (1 - \varrho\alpha)\sigma_j^z + \varrho \sigma_j^x \sigma_{j+1}^x \tag{34}$$

$$\tilde{\sigma}_j^y \tilde{\sigma}_{j+1}^y = (\varrho - \alpha^{-1}) \sum_{k=1}^{j-1} \alpha^{j-k} \sigma_k^y S_{k+1,j} \sigma_{j+1}^y + (\varrho - \alpha) \sum_{k=j+2}^{L} \alpha^{j-k} \sigma_{j+1}^x S_{j,k-1} \sigma_k^x + (1 - \varrho\alpha^{-1})\sigma_{j+1}^z + \varrho \sigma_j^y \sigma_{j+1}^y \tag{35}$$

$$\tilde{\sigma}_j^z = (\alpha^{-1} - \varrho) \sum_{k=1}^{j-1} \alpha^{j-k} \sigma_k^y S_{k+1,j-1} \sigma_j^y + (\alpha - \varrho) \sum_{k=j+1}^{L} \alpha^{j-k} \sigma_j^x S_{j+1,k-1} \sigma_k^x + \varrho \sigma_j^z \tag{36}$$

$$\tilde{\sigma}_j^y \tilde{\sigma}_{j+1}^x = (\varrho - \alpha^{-1}) \sum_{k=1}^{j-1} \alpha^{j-k} (\alpha \sigma_k^y S_{k+1,j} \sigma_{j+1}^x - \sigma_k^y S_{k+1,j-1} \sigma_j^x) + (\varrho - \alpha) \sum_{k=j+2}^{L} \alpha^{j-k} (\sigma_j^y S_{j+1,k-1} \sigma_k^x - \alpha \sigma_{j+1}^y S_{j+2,k-1} \sigma_k^x) + \Big(\varrho(\alpha + \alpha^{-1}) - 1\Big) \sigma_j^y \sigma_{j+1}^x. \tag{37}$$

Here $S_{i,k}$ denotes the Jordan–Wigner string between the sites i and k

$$S_{i,k} = \prod_{j=i}^{k} \sigma_j^z \qquad S_{i+1,i} \equiv 1 \tag{38}$$

which means that the $q \leftrightarrow \eta$ transformation converts local observables to linear combinations of strings measuring the charge between certain positions. Notice that (34)–(37) simplify in the thermodynamic limit $L \to \infty$, where $\varrho \to \alpha$ if $|\alpha| < 1$ and $\varrho \to \alpha^{-1}$ if $|\alpha| > 1$, respectively.

We have discovered the identity (12) by first noticing that the spectra of $H^{XY}(\eta, q)$ and $H^{XY}(q, \eta)$ are identical and we have computed the relations (34)–(37) by hand. Then using these results we conjectured the general structure of the transformation (16).

Let us finally check the Ising limit described above (cf equation (11)). For $q \to \infty$ and $\eta \to 0$ equations (26) and (27) reduce to

$$\tilde{\sigma}_i^z = \sigma_i^x \sigma_{i+1}^x \qquad \tilde{\sigma}_i^y \tilde{\sigma}_{i+1}^y = \sigma_{i+1}^z. \tag{39}$$

Now if we rotate $\tilde{\sigma}^x$ and $\tilde{\sigma}^y$ as in (9), we end up with

$$\tilde{\sigma}_i^z = \sigma_i^x \sigma_{i+1}^x \qquad \tilde{\sigma}_i^x \tilde{\sigma}_{i+1}^x = \sigma_{i+1}^z \tag{40}$$

and this is just the Ising duality transformation given in (6).

The $q \leftrightarrow \eta$ symmetry (29) may be interpreted physically as follows: The parameter q fixes (apart from the magnetic field) the boundary conditions of the system, while the parameter η describes the bulk anisotropy. So the exchange of q and η may be understood as a transformation which exchanges the bulk and boundary properties of the chain. An investigation of correlation functions confirms this interpretation.

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